Linear Algebra : Vector Spaces and Fields

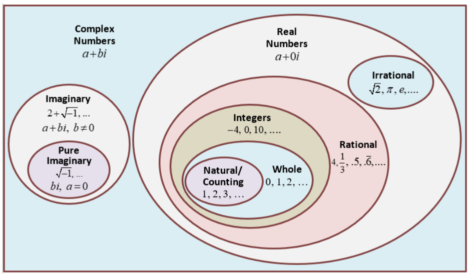
Table of Contents

1 Vector Spaces

1.1 Pre-Requisite Notations

You should already know most the **symbols for types of numbers**, but all have been included here for completeness

- $\mathbb{N}$ = Natural Numbers = $\{1,2,3, \dots\}$. Note: some courses include 0
- $\mathbb{W}$ = Whole numbers = $\{0,1,2,3, \dots\}$
- $\mathbb{Z}$ = integers = $\{\dots, -3, -2, -1, 0, 1, 2, 3 \dots\}$
- $\mathbb{Q}$ = Rational Numbers = $\left\{\frac{p}{q};\text{ p and q are integers}\right\}$
- $\mathbb{I}$ = Irrational Numbers = $\{\text{non rational numbers}\}$
- $\mathbb{R}$ = Real Numbers = $\{\text{all of the above number sets}\}$
- $\mathbb{C}$ = Complex. Numbers = $\{a + bi\text{ ; a and b are real, } i = \sqrt{-1}\}$
- $\mathbb{Z}^*$ = all integers excluding 0
- $\mathbb{Z}^+$ or $\mathbb{Z}_+$ = positive integers
- $\mathbb{R}^+$ or $\mathbb{R}_+$ = positive real numbers
- $\mathbb{Z}^\times$ or $\mathbb{Z}^*$ = integers without 0. You'll also see $\mathbb{Z} - \{0\}$ in some courses.
- $\mathbb{Z}_n$ or $\mathbb{Z}/n$ or $\mathbb{Z}/n\mathbb{Z}$ = the integers mod n = $\{0,1,2,3,4, \dots n-1\}$
- $(\mathbb{Z}_n)^\times$ = set of integers coprime to n (the integers which are invertible mod n)
  Watch out as a superscript of $\times$ can also mean omit zero



You need to first understand the following - **cartesian products and ordered pairs, $\mathbb{R}^n$ notation and tuples and function notation.**

1) Cartesian products ($A \times B$) and ordered pairs

$A \times B$ means we allocate an element of the set A and PAIR it up with an element of set B to get the form (a, b) where (a, b) is an ordered pair.

An ordered pair (a, b) respects an order and represents an element of a cartesian product. $(a, b) \neq (b, a)$ unlike with sets where $\{a, b\} = \{b, a\}$

Example:

$A = \{1,2,3\},\ B = \{a, b\}$. Find $A \times B$

$A \times B$ means the element of A comes first and the element of B comes next in the ordered pairs

$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$

2) $\mathbb{R}^n$ notation and tuples

A point is zero dimensional, it is just a location in space. From a point we can construct 1 dimensional object by stringing a infinite number of points along a particular dimension. This object is called a line.

- $\mathbb{R}$ - This is a 1-dimensional real value e.g. x **(line)**

If we string an infinite number of lines along a dimension direction perpendicular to the line we get a plane!

- $\mathbb{R}^2$ - This is a 2-dimensional real value e.g. (x, y) **(flat surface – 2D plane)**

  An example of an element in $\mathbb{R}^2$ is (1,2). The superscript of 2 is saying that we have the 2 coordinates i.e. (x, y). A 'two-dimensional real value' is also expressible as a pair of real numbers.

  Common mistake: Don't not think the superscript of 2 means you have to have a square number. The second element in our example is 2 and is not a square number and this is ok!

  $\mathbb{R}^2$ is called a Euclidean plane

If we string an infinite number of planes in either direction we get three dimensional space

- $\mathbb{R}^3$ - This is a 3-dimensional real value e.g. (x, y, z) – **space**

  $\mathbb{R}^3$ is called Euclidean space

$\mathbb{R}^n$ is a real vector space that has **n tuples** of real numbers. These are just n dimensional vector valued functions i.e. $\boldsymbol{n} \times \mathbf{1}$ **OR** $\mathbf{1} \times \boldsymbol{n}$ matrices.

So, we can say $\mathbb{R}^n$ is a vector space (which you will see later) of dimension n i.e. contains the elements $(x_1, x_2, \dots x_n)$ or $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ where $x_1, x_2, \dots x_n \in \mathbb{R}$. It is important that you realise $\mathbb{R}^n$ is just the n-tuples $(x_1, x_2, \dots x_n)$ or $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$.

You also commonly deal with complex vector spaces $\mathbb{C}^n$. This is basically the same as $\mathbb{R}^n$. It just means $x_1, x_2, \dots x_n \in \mathbb{C}$ i.e. complex numbers instead of real numbers.

3) The cartesian product $\mathbb{R}^n \times \mathbb{R}^m$

$\mathbb{R}^n \times \mathbb{R}^m$ should now make sense no that you understand the cartesian product and $\mathbb{R}^n$ notation

- $\mathbb{R} \times \mathbb{R}$

$$\mathbb{R} = (x) \text{ and } \mathbb{R} = (y)$$

More formerly we can write

$$\mathbb{R} = \{(x);\ x \in \mathbb{R}\} \text{ and } \mathbb{R} = \{(y);\ y \in \mathbb{R}\}$$

Hence the cross product is

$$\mathbb{R} \times \mathbb{R} = (x) \times (y) = \{(x, y)\}$$

Don't get confused! It might be slightly confusing to see (x) and (y).

Some students might prefer to see $\{\ldots,x,\ldots\}$ and $\{\ldots,y,\ldots\}$ so, it's clear that we aren't talking about sets of a single arbitrary value, but instead chosen values from a set

- $\mathbb{R} \times \mathbb{R}^2$

$$\mathbb{R} = (x) \text{ and } \mathbb{R}^2 = (y, z)$$

More formerly we can write

$$\mathbb{R} = \{(x)\ ;\ x \in \mathbb{R}\} \text{ and } \mathbb{R}^2 = \{(y, z)\ ;\ y, z\ \mathbb{R}\}$$

$$\mathbb{R} \times \mathbb{R}^2 = (x) \times (y, z) = \{(x, y, z)\}$$

- $\mathbb{R}^2 \times \mathbb{R}^3$

Let's use subscripts now since we have more letters

$$\mathbb{R}^2 = (x_1, x_2) \text{ and } \mathbb{R}^3 = (x_3, x_4, x_5)$$

$$\mathbb{R}^2 \times \mathbb{R}^3 = (x_1, x_2) \times (x_3, x_4, x_5) = \{(x_1, x_2), (x_3, x_4, x_5)\}$$

So, we have 2-tuples and 3-tuples

But this is the same as 5 tuples

$$\mathbb{R}^2 \times \mathbb{R}^3 = (x_1, x_2) \times (x_3, x_4, x_5) = \{(x_1, x_2, x_3, x_4, x_5)\} = \mathbb{R}^5$$

More formerly we can write

$$\mathbb{R}^2 = \{(x_1, x_2)\ ;\ x_1, x_2 \in \mathbb{R}\} \text{ and } \mathbb{R}^3 = \{(x_3, x_4, x_5)\ ;\ x_3, x_2, x_3 \in \mathbb{R}\}$$

$$\mathbb{R}^2 \times \mathbb{R}^3 = \{\ (x_1, x_2), (x_3, x_4, x_5); (x_1, x_2) \in \mathbb{R}^2, (x_3, x_4, x_5) \in \mathbb{R}^3\ \}$$

This is the same as 5 tuples

$$\{(x_1, x_2, x_3, x_4, x_5);\ x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}\} = \mathbb{R}^5$$

Notice how $\mathbb{R}^2 \times \mathbb{R}^3 = \mathbb{R}^5$

Now it should make sense that i**n general we write** $\mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$

- $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$

$$\mathbb{R} = (x_1) \text{ and } \mathbb{R} = (x_2) \text{ and } \mathbb{R} = (x_3)$$

  - There are 3 ways we can write this

$$\{((x_1, x_2), x_3);\ (x_1, x_2) \in \mathbb{R}^2,\ x_3 \in \mathbb{R}\}$$
or
$$\{((x_1), (x_2, x_3));\ x_1 \in \mathbb{R}\ (x_2, x_3) \in \mathbb{R}^2\}$$
or
$$\{(x_1, x_2, x_3);\ x_1, x_2, x_3 \in \mathbb{R}\}$$

So, it should make sense that $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ can be written as any of $\mathbb{R}^2 \times \mathbb{R}$ OR $\mathbb{R} \times \mathbb{R}^2$ OR $\mathbb{R}^3$

Note for later on: T:$\mathbb{R}^n \times \mathbb{R}^m$ can sometimes be used to denote a $m \times n$ matrix.

4) **Functions and mappings $f: A \to B$**

$f: A \to B$ (a function that takes a$\in A$ from the domain and returns b$\in B$ from the codomain)

We can see this written in lots of ways of which some look confusing.

$f: x \to x^2$	$f: \mathbb{N} \to \mathbb{N},\ x \mapsto 2x$	$f: \mathbb{R}^2 \to \mathbb{R},\ (x,y) \mapsto x^2 + y^2$
This means $f(x) = x^2$	This means $f(x) = 2x$	This means $f(x,y) = x^2 + y^2$
Numerical example: Let's take $x = 2$ $f(2) = 4$ So f maps 2 to 4	Numerical example: Let's take $x = 1$ $f(1) = 2(1) = 2$ So f maps 1 to 2 where 1$\in \mathbb{N}$, 2$\in \mathbb{N}$	Numerical example: Let's take $x = 1$ and $y = 2$ $f(1,2) = 1^2 + 2^2 = 5$ so f maps (1,2) to 5 where (1,2) $\in \mathbb{R}^2$ and 5 $\in \mathbb{R}$

Now you should be in a position to understand $f: \mathbb{R}^n \mapsto \mathbb{R}^m$

$$(x_1, x_2, \dots x_n) \mapsto (x_1, x_2, \dots x_m) \text{ or } \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

So,

$$f(x_1, x_2, \dots x_n) \mapsto (f_1(x_1, x_2, \dots x_n), \dots, f_m(x_1, x_2, \dots x_n))$$

Important: This notation means for any n dimensional vector, the function returns a m dimensional vector.

The Cartesian product notation $\mathbb{R}^n \times \mathbb{R}^m$ often comes up with functions/mappings. The function can send your domain to anything.

- $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ (x,y) \mapsto xy$

  This means $f\big((x,y)\big) \mapsto xy$

  Let's take a numerical example

  $$f(2,3) = 6$$

  where $(2,3) \in \mathbb{R} \times \mathbb{R}$ and $6 \in \mathbb{R}$

- $g: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R},\ \left(\binom{x}{y}, \binom{x'}{y'}\right) \mapsto 2xx' + \frac{3}{2}xy' + \frac{3}{2}x'y$

  This means $f\left(\binom{x}{y}, \binom{x'}{y'}\right) \mapsto 2xx' + \frac{3}{2}xy' + \frac{3}{2}x'y$

  Let's take a numerical example

  $$f\left(\binom{1}{3}, \binom{2}{5}\right) = 4 + \frac{15}{2} + 9 = 20.5$$

  where $\left(\binom{1}{3}, \binom{2}{5}\right) \in \mathbb{R}^2 \times \mathbb{R}^2$ *and* $20.5 \in \mathbb{R}$

Considere $T: \mathbb{R}^2 \to \mathbb{R}^3$

1.2 Pre-Requisite Definitions

We still need a few more definitions before defining vector spaces. Subtraction is the same as adding negatives and division is the same thing as multiplying by fractions, so this is the reason that **we only need to worry about addition and multiplication for fields** (since subtraction can be defined as the equivalent of addition and division can be defined as the equivalent of division). Therefore subtraction and division are not separate composition laws and we **do not need to think about subtraction or division**.

You need to make sure you understand the following 6 definitions - identity element, inverse element, closure, commutative, associate and distributive:

1) Identity element (aka neutral element) - "**Does nothing**". It leaves an element of a set unchanged when combined with it

Doing nothing with addition means to add 0 hence the identity under addition (additive identity) is 0. The additive identity states that if a number is added to zero it will give the number itself as a resultant. So, under addition, 0 is the identity element.

Doing nothing with multiplication means to multiply by 1 hence the identity (multiplicative identity) under multiplication is 1. The multiplicative identity states that if a number is multiplied by 1 the resultant will be the number itself. So, under multiplication, 1 is the identity element.

Additive Identity (0)	**Multiplicative Identity (1)**
0 is the identity element under addition For example consider the number 77 $77 + 0 = 77$ Formerly we write this as: For an arbitrary a $a + 0 = 0 + a$ adding 0 leaves a unchanged, so identity element of $+$ is 0	1 is the identity element under multiplication For example consider the number 77 $77 \times 1 = 77$ Formerly we write this as For an arbitrary a $a \times 1 = a = 1 \times a$ multiplying by 1 leaves a unchanged, so identity element of $\times$ is 1

2) Inverse - "**Undoes**". An element that **gives the identity** when composed with it. It generalises the concept of negation and reciprocation.

Additive Inverse (negative)	**Multiplicative Inverse (reciprocal)**
The negative version of the number is the inverse element under addition For example consider the number 77 $77 + (-77) = 0$ Formerly we write this as: For an arbitrary a $a + (-a) = 0$ Subtracting a gives the identity element 0, so $-$a is the inverse of a	The reciprocal of the number is the inverse element under multiplication 1 is the identity element under multiplication For example consider the number 77 $77 \times \frac{1}{77} = 0$ Formerly we write this as For an arbitrary a $a \times \frac{1}{a} = 1$ Multiplying by $\frac{1}{a}$ a gives the identity element 1, so $\frac{1}{a}$ is inverse of a under multiplication

3) Closure (Closed): Closure is when an operation (such as "adding") on members of a set (such as "real numbers") always makes a member of the same set i.e. the **result of the operation (additional, subtraction, multiplication or division) stays in the same set i.e. is still a member of the set**

Real number example
When we add two real numbers we get another real number

$$3.1 + 0.5 = 3.6$$

This is always true, so real numbers are closed under addition

Whole number example
When we subtract two whole numbers might not make a whole number

$$4 - 9 = -5$$

-5 is not a whole number since whole numbers cannot be negative. Whole numbers are not closed under subtraction

Odd number example
Odd numbers $\{..., -3, -1, 1, 3, ...\}$
Is the set of odd numbers closed under the simple operations $+ \ - \times \div$?

- Adding? 3 + 7 = 10 but 10 is even, not odd, so no odd numbers are NOT closed under addition
- Subtracting? 11 − 3 = 8 but 8 is even, not odd, so no odd numbers are closed under subtraction
- Multiplying? 5 × 7 = 35 yes ... in fact multiplying odd numbers always produces odd numbers, so odd numbers are closed under multiplication
- ○ Dividing? 33/3 = 11 which looks good! But try 33/5 = 6.6 which is not odd, so no

4) Commutative "Commute, move around"**.** If we re- order numbers we still get the same answer.

Addition: $a + b = b + a$
$6 + 3 = 3 + 6$

Multiplication: $a \times b = b \times a$
$2 \times 4 = 4 \times 2$

5) Associative "Associate or group"**:** If re-group get the same answer, meaning it doesn't matter which number we calculate first

Addition: $(a + b) + c = a + (b + c)$
$(6 + 3) + 4 = 6 + (3 + 4)$

Multiplication: $(a \times b) \times c = a \times (b \times c)$
$(2 \times 4) \times 3 = 2 \times (4 \times 3)$

6) Distributive: "When you distribute something, you give pieces of it to many different people such as handing out papers in class." In math, people usually talk about the distributive property of one operation over another (most common is multiplication over addition). It says that when a number is multiplied by the sum of two other numbers, the first number can be handed out or distributed to both of those two numbers and multiplied by each of them separately. **This is the rule that connects addition and multiplication.**

Multiplication: $a \times (b + c) = ab + ac$

$(2 \times 4) \times 3 = 2 \times (4 \times 3)$

1.3 Topics Link

Ideally one should have a basic knowledge of groups and fields before starting vector spaces and fields and groups are usually taught before vector spaces for this reason. This is because in order to define a vector space we need to know what a field is because a vector space is defined relative to a field. However, in order for ease of explanation and being able to grasp the topic, I will first define a vector space and then explain where the fields come into it and then cover groups after.

Vector spaces are a really good place to start since they introduce the definitions/properties of associative, commutative, distributive, identity, inverses and closure which are vital in groups and fields!

Being good at vector spaces though is actually making sure you totally understand these properties and basically ticking them off one by one. Vector spaces, fields, groups are simply just a case of checking whether these properties hold!

1.4 Intuitive Definition

Most students find vector spaces very hard as the concepts are very abstract (you no longer have something to solve or compute). This is compounded by the fact that most professors most teachers don't take the time to simplify these things and they just go on as if you already know what a vector space is. They just quickly define it, but the concept is not clear! This section will concentrate on what vector spaces really mean and what you need to do just to move on.

So many students will tell you even after their degree that they never got a conclusive answer to what a vector space or a vector was that didn't reference the other and still struggle to explain what a vector space is. Think of a vector space is just a place/object where we take a bunch of different direction vectors (like x, y, and z.) and then work out all the possible places you can get by moving in multiples of those vectors. Think of vectors as arrows. Vectors are "the same" as arrows, given a set of axes.

Let's get slightly more in detail now. A vector space is a space/environment. But mathematical speaking, a vector space is a set of vectors that can do two essential things:

- addition – they can add to each other (you can take one thing and add it to the other one in that set).
- multiplication – they can be stretched by scaling (scalar multiplication). We can make them bigger or small.
  Important to note: We cannot multiply the vector together by each other, but we can multiply by a number.

And there is an **extra property** (known as **closure**) for each pay attention to closure as it will come up again in groups!

- **For** addition - any set of stuff you can add together such that when you add them together the addition/result will behave as if it was in the set already. We called this closed under addition.
- **For** multiplication - if you pick anything in that set (an element of the set) and you multiply it by a scalar quantity (for example multiply by real number - avoid zero since it has special characteristics makes things disappear). The element will either get bigger or smaller, but it is still that element, it is just a smaller or bigger version of it hence it behaves as if it was in the set already. We call closed under multiplication.

Why do we only consider the 2 operations addition and multiplication above?
Recall from earlier than subtraction is the same as adding negatives ($3-5$ is the same as $3+-5$), so instead of having to use addition and subtraction, we only need addition! Also recall that division is the same as multiplying by reciprocals $\frac{3}{5}$ is the same as $\left(3\times\frac{1}{5}\right)$, so instead of having to use multiplication and division, we only need multiplication!

Hence it should make sense that we only need addition and multiplication and that is the reason they are the only 2 operations mentioned above.

And again, let's get slightly more formal with the definition:
A vector space is a set of vectors for which addition and scalar multiplication are defined (can perform addition and scalar multiplication) such that if $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in the space and k is a scalar (like 1, 2, 3,4, $\frac{1}{2}$, -7) then $\boldsymbol{u}+\boldsymbol{v}$ (result of adding 2 vectors) is in the space and $k\boldsymbol{u}$ (result of multiplying a vector by a scalar, the new of magnified or contracted vector) is also in the space.

The very best way to get an intuitive understanding from that point is just to see a lot of vector spaces. Let's look at some examples:

- **Set of polynomials of first degree**

$$\boldsymbol{u}=x+1 \text{ and } \boldsymbol{v}=2x-3$$

  Can we add $\boldsymbol{u}$ and $\boldsymbol{v}$? Yes!
  If we add them we get $\boldsymbol{u}+\boldsymbol{v}=3x-2$
  $3x-2$ is a polynomial of first degree hence also in the vectors space

  Can we multiply by a scalar? Yes!
  If we multiply $\boldsymbol{u}$ by 5 we get $5\boldsymbol{u}=5(x+1)=5x+5$
  $5x+5$ is a polynomial of first degree hence also in the vector space

  $\boldsymbol{uv}=(x+1)(2x-3)=2x^2-x-3$ which is not a member of vector of the vector space. We have a polynomial of second degree now!

  This highlights how we cannot multiply the vectors together, we can only multiply by a scalar in vector spaces!

- **Set of 2x2 matrices**

$$\boldsymbol{u}=\begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix} \text{ and } \boldsymbol{v}=\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}$$

  This is a vector space. Why?
  You can add the two matrices together and you can multiply the matrix by a scalar.
  If you add 2 matrices you end up with another 2x2 matrix and if you multiply by a scalar you end up with another 2x2 matrix

- **Derivatives**

$$\boldsymbol{u}=\frac{d}{dx}f(x) \text{ and } \boldsymbol{v}=\frac{d}{dx}g(x)$$

  This is just the derivative of all functions that are differentiable.

$$\boldsymbol{u}=\frac{d}{dx}\sin x \text{ and } \boldsymbol{v}=\frac{d}{dx}\cos x$$

  You can add derivatives

$$\frac{d}{dx}(\sin x+\cos x)$$

  Pick a differentiable function. When you do $\frac{d}{dx}(\sin x+\cos x)$ you get another function that is differentiable!

- **Integrals** – this is the same as above for functions that can be integrated

- **Numbers** (scalars) - You can add 1 to 2 and you can multiply 1 by 2 and whatever you get will always be a number

- The **real numbers** $\mathbb{R}$ are a vector space
  .
- The **complex numbers** $\mathbb{C}$ are a vector space

- The **Euclidean space and planes** $\mathbb{R}^n$ and $\mathbb{C}^n$ are vector spaces (recall these are n tuples)

So, a vector space is a set of "things" that you can "**add together**" and "**multiply** by numbers" such that certain nice properties hold. The best way to get an intuitive understanding is just to see a lot of vector spaces. We can add two elements (vectors) togethers to get another vector, and we can scale each vector by a number (a scalar) to get another vector.
A vector space defines a "club" of mathematical things that obey certain properties and interact with each other in certain ways. In this context, the usual pointy arrow types of vectors are certainly members of a vector space, but matrices can be considered "vectors", polynomials can be "vectors", functions can be "vectors", and so on. The different examples above should have made this clear.

Let's look at vector spaces with our familiar $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$ notation. Vector spaces are any space made up of vectors meaning that vectors are any elements of a vector space. You should be starting to get a good picture in your head of what vector spaces are (ie.$\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$, etc) and we also know that they have many properties. A few of the most important are that vector spaces are closed both under addition and scalar multiplication. What does that mean? Being **closed under addition** means that if we took any vectors x_1 and x_2 and added them together, their sum would also be in that vector space. Focusing on the examples below will really help solidify the notion of a vector space.

ex. Take $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\begin{pmatrix}3\\1\\2\end{pmatrix}$. Both vectors belong to $\mathbb{R}^3$.

- Their sum, which is $\begin{pmatrix}4\\3\\5\end{pmatrix}$ is also a member of $\mathbb{R}^3$. Being closed under scalar multiplication means that vectors in a vector space, when multiplied by a scalar (any real number), it still belongs to the same vector space.
- Consider $\begin{pmatrix}1\\4\\3\end{pmatrix}$. If I multiply the vector by a scalar, say, 10, I will get $\begin{pmatrix}10\\40\\30\end{pmatrix}$ which is still in $\mathbb{R}^3$

$\mathbb{C}^3$ would be the exact same except same except there are complex numbers within the vectors.

1.5 A More Formal Definition

Now you should be starting to get a general idea of what vector spaces are, so let's try to formalise them even more.
.
So far, we have defined addition and scalar multiplication and we these are often described as being 'well defined' for addition and well defined for scalar multiplication. Addition and scalar multiplication are the most essential, but we need extra properties. The process of whenever you add or multiply by a scalar must have certain behaviours. These things that define their behaviours/properties/characteristics are called the **axioms** of a vector space. What we have done so far can be called the well-defined axioms, but there are more!

What are these axioms? This is where students begin to struggle. You need to get used to no longer memorising things that you understand. You have to memorise every single axiom and need to know everything about the axioms of a vector space. Maybe not in the same order as your friend but you have to know everything about the axioms of a vector space! Make sure you can remember them and perform them anytime you are asked to verify a vector space.

Here we go!!! Let V be vector space and $\mathbb{F}$ be a field of SCALARS

(Note: We will look in detail what a field $\mathbb{F}$ is later on. Just see where appears for a vector space and how we use it. So far we have used real numbers (1,2, -7 etc) but sometimes a field of scalars is complex numbers or imaginary numbers. We have used word scalars since sometimes the numbers are not real, but for now we will just talk about real numbers on the number line, they don't have to be rational, they just have to be real. Just realise that a vector space is defined over a field, from which the scalars are drawn)

If $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$ and $a, b \in \mathbb{F}$ then
(make sure you understand $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ vectors and $\boldsymbol{a}, \boldsymbol{b}$ scalars)

- **Axiom 1 (commutative property of addition)**:
  In English: If we switch the order of addition get same answer.
  $$\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$$
  For example, $1 + 2 = 2 + 1$ and this has to be true for any vector space. Also consider adding polynomials and switch the order it doesn't matter we get the same answer.

- **Axiom 2 (associative property of addition)**
  In English: The order in which you add does not matter
  $$\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$$

- **Axiom 3 (additive identity)** - existence of the zero vector. Adding zero leaves things unchanged!
  In English: There has to be some vector in the vector space such that when add that vector to any other vector it doesn't change the other vector
  There is a zero vector $\boldsymbol{0}$ such that $\boldsymbol{u} + \boldsymbol{0} = \boldsymbol{0} + \boldsymbol{u} = \boldsymbol{u}$
  $\boldsymbol{0}$ is the zero vector and the additive identity
  For example, $0 + 5$ is always 5, it doesn't change

  Axiom 4 (**additive inverse**) - What you must add to something to make it 0
  In English: For every vector in vector space has another guy such that when you combine the two of them gives you zero additive inverse

  For every $\boldsymbol{u} \in V$, there is a $-\boldsymbol{u} \in V$ such that $\boldsymbol{u} + (-\boldsymbol{u}) = \boldsymbol{0}$
  $-\boldsymbol{u}$ is the additive inverse
  Note: some courses use $\boldsymbol{u}'$ instead of $-\boldsymbol{u}$

  For example, 5. It is obvious that -5 is the additive inverse of 5

- **Axiom 5 (multiplicative identity)** – Multiplying by 1 leaves things unchanged.

  In English: What is the scalar used to multiply and doesn't change anything? It is 1!
  For every $\boldsymbol{u} \in V$, $1\boldsymbol{u} = \boldsymbol{u}$
  1 is multiplicative identit

- **Axiom 6 (associative property of multiplication)**
  In English: We can switch the order of the scalars
  $$(ab)\boldsymbol{u} = a(b\boldsymbol{u})$$
  Remember a and b are scalars. This is obvious and common sense, but sometimes it is the only way out of whatever you are stuck on and realise you can actually switch a and b!

- **Axiom 7: (distributivity of addition over multiplication)**
  In English: Instead of just multiplying, what about adding? We can distribute the addition of scalars over multiplication. This is the link between addition and multiplication!
  $$c(\boldsymbol{u} + \boldsymbol{v}) = c\,\boldsymbol{u} + c\,\boldsymbol{v}$$
  $$(c + d)\boldsymbol{u} = c\,\boldsymbol{u} + d\,\boldsymbol{u}$$

Take note (important!!!):
We **don't need** a **multiplicative identity** for vectors **nor commutativity for multiplication** here since we cannot multiply vectors together in vector spaces! Also, we don't need a multiplicative identity for scalars as an axiom since a vector space is defined over a field, from which the scalars are drawn. Since F is a field, all nonzero elements are guaranteed to have inverses. There is no need to repeat that in the definition of vector space. A

1.6 The Most Formal Definition

Definition (well defined property):
A nonempty set V of objects (called vectors) which are defined by two operations - addition and multiplication by scalars (real numbers). It is important to realize that a vector space consists of four entities: a set of vectors, a set of scalars (real numbers) and two operations (+ and ×)

10 Axioms - I have included closure as an axiom as lots of lecturers do. I introduced closure separately first to give you intuition of vector spaces.

Consider vectors u, v and w ∈ V and scalars c and d. We need the following axioms for Addition and Multiplication to hold. The red text below highlights the need for field $\mathbb{F}$ which we will now encounter.

Addition:

1) **Closure:**
$$\boldsymbol{u}+\boldsymbol{v} \in V$$
2) **Commutative**:
$$\boldsymbol{u}+\boldsymbol{v} = \boldsymbol{v}+\boldsymbol{u}$$
3) **Associative:**
$$(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w} = \boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$$
4) **Identity:**
There is vector $\mathbf{0} \in V$ such that $\boldsymbol{u}+\mathbf{0} = \boldsymbol{u}$
5) **Additive Inverse:**
For each $\boldsymbol{u} \in V$, there is a vector $-\boldsymbol{u} \in V$ whereby $\boldsymbol{u}+(-\boldsymbol{u}) = \mathbf{0}$

Multiplication:

6) **Closure:**
$$c\,\boldsymbol{u} \in V$$
7) **Associative:**
$$(cd)\boldsymbol{u} = c(d\boldsymbol{u})$$
8) **Identity:**
$$1\boldsymbol{u} = \boldsymbol{u}$$
9) **Distributive:**
$$c(\boldsymbol{u}+\boldsymbol{v}) = c\,\boldsymbol{u} + c\,\boldsymbol{v}$$
$$(c + d)\boldsymbol{u} = c\,\boldsymbol{u} + d\,\boldsymbol{u}$$

1.7 Vector Spaces Summary

1. Definition of a Vector Space
A vector space is a collection of obects (vectors) that can be added together and multiplied by scalars (elements from a field) while satisfying certain axioms i.e. certain properties hold (like associativity, commutativity of addition, identities, inverses and distributivity of scalar multiplication, etc.).

That's it. That's the entire concept. If you have a collection of things that you can add or scale, and the result is another object in your collection, that collection is a vector space. The list of axioms is just nailing down what exactly "can be added together, or scaled by a constant" means, but they're exactly the rules you would expect addition and multiplication to have.

2. Operations in Vector Spaces
Recall that subtraction is the same as adding negatives and division is the same as multiplying by reciprocals so we only need to worry about addition and multiplication, but only multiplication by scalars between the vectors in the set!
In a vector space, the primary operations are:
Vector Addition: Combining two vectors to produce another vector.
Scalar Multiplication: Multiplying a vector by a scalar (a number from a field).

3. Lack of Multiplicative Structure
No Multiplication of Vectors: In a vector space, there isn't a defined operation that allows for the multiplication of two vectors to produce another vector (unless you're considering specific types of vector spaces, like those associated with inner products). Therefore, the notion of a "multiplicative inverse" for a vector itself doesn't exist.

4. Scalar Multiplicative Inverses
Scalars Have Inverses: While vectors do not have multiplicative inverses, the scalars from the field do. For any non-zero scalar a, there exists a multiplicative inverse a^{-1} such that $a \times a^{-1}$=1. This property is essential for the scalar multiplication operation in vector spaces.

5. Conclusion
In summary, vector spaces do not have multiplicative inverses for vectors because:
- vectors are not multiplied in a way that produces other vectors within the framework of standard vector space operations.
- The focus in vector spaces is on addition and scalar multiplication, not on multiplicative structures involving vectors themselves

1.8 Vector Subspaces

Now that you know what a vector space is, you are ready to define what a subspace is. Strictly speaking, a Subspace is a Vector Space included in another larger Vector Space. Therefore, all properties of a Vector Space, such as being closed under addition and scalar multiplication still hold true when applied to the Subspace.

To show something is a subspace you must check the three conditions: it contains the zero vector, it is closed under addition, it is closed under scalar multiplication.

We all know $\mathbb{R}^3$ is a Vector Space. It satisfies all the properties including being closed under addition and scalar multiplication. Consider the set of all vectors S=$\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$, such that x and y are real numbers. This is also a vector space because all the conditions of a vector space are satisfied, including the important conditions of being closed under addition and scalar multiplication.

Consider the vectors $\begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}$ which are both contained in S. If we add them together, we get $\begin{pmatrix} 6 \\ 6 \\ 0 \end{pmatrix}$ which is still in S. We can multiply each one by the scalar $\frac{1}{2}$ and get $\begin{pmatrix} \frac{1}{2} \\ 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \frac{5}{2} \\ 1 \\ 0 \end{pmatrix}$ which are both still in S.

So we see that S is a Vector Space, but it is important to notice that all of S is contained in $\mathbb{R}^3$. By this, I mean any vector in S can also be found in $\mathbb{R}^3$. Therefore, S is a SUBSPACE of $\mathbb{R}^3$

Other examples of Sub Spaces:

- Lower powers of $\mathbb{R}$ are subspaces of higher powers of $\mathbb{R}$
- The line defined by the equation $y = 2x$, also defined by the vector definition $\begin{pmatrix} t \\ 2t \end{pmatrix}$ is a subspace of $\mathbb{R}^2$
- The plane $z = -2x$, otherwise known as $\begin{pmatrix} t \\ 0 \\ -2t \end{pmatrix}$ is a subspace of $\mathbb{R}^3$

1.8.1 Determining Whether A Subset Is A Subspace

There are two very important notions of a vector space, and they will end up being very important in defining a sub space.

If given a space and asked whether or not it is a sub space of another vector space, there is a very simple test you can perform to answer this question. There are only two things to show: The subspace test to test whether or not S is a subspace of some vector space $\mathbb{R}^n$ you must check two things:

1) if $\boldsymbol{s_1}$ and $\boldsymbol{s_2}$ are vectors in S, their sum must also be in S
2) if $\boldsymbol{s}$ is a vector in S and k is a scalar, $k\boldsymbol{s}$ must also be in S

In other words, to test if a set is a subspace of a vector space, you only need to check if it closed under addition and scalar multiplication. Easy!
Note: Containing the zero vector is implied by being closed under scalar multiplication.

2 Fields

2.1 Notation

$\mathbb{F}$ is the notation used to denote a field.

2.2 Intuitive Definition – vector spaces versus fields

Recall from the vectors spaces earlier we used scalars to define what the vector space was 'over'. So the need for a field (scalars) should at least make sense at this point.

We saw earlier why we have the need for a field. In vector spaces we can ONLY add vectors, we cannot multiply vectors. In a field, you can add things and multiply them. So, by using fields we can multiply a vector with a scalar (element of a field) to get a new vector.

I will now repeat this to make sure understand that any two elements can be multiplied in a field, but it is not allowed in a vector space - only scalar multiplication is allowed whereby the scalars are from the field. So, vector spaces come equipped with an action by a field. In more precise words: vector spaces are modules over a field.

- $\mathbb{C}$ is a vector space over $\mathbb{R}$ since every complex number is uniquely expressible in the form $a + bi$ with $a, b \in \mathbb{R}$.
  Other way round $\mathbb{R}$ is a vector space over $\mathbb{C}$ not so obvious, but you may be asked to prove it one day ☺
- $\mathbb{R}^n$ is a vector space over $\mathbb{R}$
- $\mathbb{C}^n$ is a vector space over $\mathbb{C}$

Recall that a vector space is just a place/object where we take a bunch of different direction vectors (like x, y, and z) and then work out all the possible places you can get by moving in multiples of those vectors. And a field is like taking a group of vector like objects, a big space that represents all the places you can move by combining those vectors, and also having two different operations you can do to those vectors.

I could go very formal now, but I don't think that will be particularly helpful. Let me try a few two-sentence explanations and see if any of them 'stick' or resonate.

- A vector space is something like actual space - a bunch of points and a vector field is an association of a vector with every point in actual space
- A vector space you draw as a coordinate system. A vector field you draw as a bunch of vectors all over your plane/space/etc.
- Complex numbers are basically just 2D vectors. We can generalise to allow angles of a particle at the point in space too.
- A vector space is a set of possible vectors and a vector field is, loosely speaking, a map from some set into a vector space.

A very loose way to think about this is that a vector field is very "demonstrable", in that it's just a spray of vectors all over the place. Weather maps that use arrows to indicate wind speed direction are displays of vector spaces. Diagrams of an electric field where arrows from the direction and magnitude of current are displays of vector fields. If you take a handful of needles and toss them all over the floor, you've visualized a vector field. A vector field assigns a vector to any point in space. So, vector fields are very visual, and vector spaces are very 'big-brain things'.

2.3 Loose Definition

Loosely speaking, a field is a set (notation $\mathbb{F}$) of elements with 4 operations where you can freely add, subtract, multiply and divide and where the usual rules of arithmetic apply for all the elements

- Associate under addition
$$a + (b + c) = (a + b) + c$$
- Associate under multiplication:
$$a(bc) = (ab)c$$
- Commutative under addition
$$a + b = b + a$$
- Commutative under multiplication
$$ab = ba$$
- Distributive under multiplication (this property connects/links addition and multiplication)
$$a(b + c) = ab + ac$$
  Note: We don't need to write $(b + c) \times a = ba + ac$ since we already mentioned commutative under multiplication

So, $\mathbb{F}$ is a set on which addition, subtraction, multiplication and division are defined and behave as the corresponding operations on rational and real numbers do.

Recall that **subtraction is the same as adding negatives and division is the same as multiplying by fractions**. So, you will hear the definition of a field as being the **set of elements with 2 operations (addition and multiplication)** and that both operations are connected by the distributive properties. This means we can update the definition of a field in our mind as a set of elements where you can freely add and multiply (but we know this also means subtraction and division).

2.4 Examples

This will help you grasp the definition of a field better. Common examples to look at are the natural numbers, integers, rational numbers, real numbers and complex numbers

- $\mathbb{N}$: Not a field. They do not even possess additive inverses.
  For example, there is no additive inverse for 2. We can't have -2. (this will make sense once you read section 1.2)
- $\mathbb{Z}$: Not a field since
  For example, there is no multiplicative inverse for example 2. For example we can't have $\frac{1}{2}$ since $\frac{1}{2} \notin \mathbb{Z}$
- $\mathbb{Q}$: This is a field. This is the set of numbers that can be a fraction $\frac{p}{q}$ where p is an integer and q is a natural number, $q \neq 0$
  For example, additive inverse of $\frac{3}{5}$ is $-\frac{3}{5}$ also a rational number. Multiplicative inverse of $\frac{3}{5}$ is $\frac{5}{3}$ also a rational number.

- $\mathbb{R}$: This is a field. We know the reals contain all of the 3 number sets above, so this is an obvious answer that it is not a field since
- $\mathbb{C}$: This is a field. These are a sort of $\mathbb{R}^2$ ordered pair
- $\mathbb{R}[x]$: The collection of Polynomials with real number coefficients
  You can freely +, -, x these polynomials but ÷ is a problem
  Reciprocal of $x+1$ is $1/x+1$ which is not a polynomial
  Hence not a field
  But we can make it into a field if expand collection of polynomials to include rational functions . These are fractions where the numerator and denominator are polynomials. If we can do this then we can divide and everything is commutative so we have a field, the field of rational functions

There are lots of other fields, even ones with only a finite number of elements called finite fields.

This section is written to a basic simplified level, so don't feel confused if/when your lecturers broaden the definitions. Your linear algebra lecturer will most likely just talk about defining some sort of addition and multiplication operation and having elements in the field. Field elements don't even need to be numbers at all. So, most of the fields you will encounter through the first year of university will just use the addition and multiplication you're familiar with. Be aware that you may be hit with extensions on this topic, but don't worry for now.

A field is a mathematical object that on one hand is a relatively simple generalization of the ideas behind groups (don't worry if you haven't come across groups yet), but on the other will allow us to understand a variety of beautiful mathematical concepts and applications. Your lecturer should go into further detail with fields when you first learn groups.

2.5 A More Advanced Intuitive Definition

A vector space is a space of vectors, i.e. each element is a vector. A vector field is, at its core, a function between some space and some vector space, so every point in our base space (see vector base section) has a vector assigned to it. A good example would be wind direction maps you see on weather reports. Technically, positions on a map are also at least two-dimensional, and therefore vectors themselves, making this a bit more confusing, but for now this is beyond the scope that you need for your first year studying maths at university, so don't worry.

Imagine that you are riding a rollercoaster (represented by a flat stripe) and are allowed to move your head around in either direction (360 degrees :) The directions in which you may be looking and the distance to which you focus your eyes represent a vector space. At every point, there is a vector space of possible configurations, and these configurations slide along the rollercoaster. It's OK to gradually turn your head, which will result in a vector field --- at every point of the rollercoaster ride, you'd be looking somewhere (represented by a tangent vector). Now imagine a rollercoaster on a sphere or a torus. If the creature riding the rollercoaster has two heads (each with a pair of eyes), that would give a more complicated vector bundle and higher-dimensional vector fields which you will deal with much later on, so do not worry too much about this last sentence for now.

More formerly we say:
vector spaces and fields are two distinct algebraic structures, with the main difference being that a field allows for multiplication of its elements
You can multiply the elements of a field together. Generally you cannot do this with a vector space, but you can multiply elements of a vector space by elements from the underlying field. In this way, every field is a vector space over itself. In fact, every field is a vector space over any of its subfields.

Also note that In the context of vector spaces, the concept of multiplicative inverses is not applicable in the same way it is in fields (or rings for when you get to that).

2.6 More Formal Definition

A field is a set of elements {…..} with 2 operations $+$ and $\times$.
It satisfies the following properties

- **Closure**
  $+: \forall x, y \in \mathbb{F}, x+y \in \mathbb{F}$ i.e if you add any 2 elements of the set you get another element of the set
  $\times: \forall x, y \in \mathbb{F}, xy \in \mathbb{F}$ i.e if you multiply any 2 elements of the set you get another element of the set
  with 2 operations $+$ and $\times$
- **Associative**
  $+: (x+y)+z = x+(y+z) \ \ \forall x, y, z \in \mathbb{F}$
  $\times: (xy)z = x(yz) \ \ \forall x, y, z \in \mathbb{F}$
- Commutative: under addition and multiplication (if you omit zero since can't divide by 0)
  $+: x+y = y+x \ \ \forall x, y, \in \mathbb{F}$
  $\times: xy = yx \ \ \forall x, y \in \mathbb{F}$
  Table symmetrical about diagonal

- **Identity**
  $+$: $\exists 0 \in \mathbb{F}$, s.t. $0 + x = x \; \forall x \in \mathbb{F}$ (table row and column item is itself)
  $\times$: $\exists 1 \neq 0 \in \mathbb{F}$, s.t. $1 \times x = x \;\; \forall x \in \mathbb{F}$
  Sometimes you will see the following notation We write $\mathbb{F} = (\mathbb{F}, +, 0, \times, 1)$
- **Inverse**
  $+$: $\forall x \in \mathbb{F}$, $\exists -x \in \mathbb{F}$ s.t. $x + (-x) = 0$
  $\times$: $\forall x \neq 0 \in \mathbb{F}$, $\exists\, x^{-1} \in \mathbb{F}$ s.t. $x \times x^{-1} = 1$
- **Distributive property:** This connects addition and multiplication
  $\forall x, y, z \in \mathbb{F}, a(b + c) = (a \times b) + (a \times c)$